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1992 J. Phys. A: Math. Gen. 25 L1033

(http://iopscience.iop.org/0305-4470/25/17/003)

J. Phys. A: Math. Gen. 25 (1992) L1033–L1038. Printed in the UK

LETTER TO THE EDITOR

A new integrable equation in the (2+1)-dimension-dispersionless limit of the BKP equation

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Received 18 March 1992, in final form 15 June 1992

Abstract. A new integrable equation is obtained as the dispersionless limit of the BKP equation. It is then studied in the framework of Kodama's approach. The new equation also turns out to be completely integrable with an infinite number of conservation laws. The method of hodograph transformation is applied to obtain exact solutions in various cases.

Dispersion limits of integrable systems were initially studied Lax and Levermore [1]. The KdV equation was studied and some excellent new properties were obtained [2]. In a later communication [3], the dispersionless limit of the KP equation was exhaustiively studied by Kodama. There, he showed that the Zabolotskaya–Khohlov equation [4] could be deduced from the dispersionless limit of the KP equation. Exact solutions of this new equations were then obtained via the hodographic transformation. Here in this letter we have studied the dispersionless limit of the BKP equation and have obtained a new equation which is also integrable. Some exact solutions of this new equation are obtained by making recourse to the methodology of Kodama. It is also proved that an infinite number of conservation laws also exist for this new nonlinear system.

Let L denote the pseudodifferential operator [5]

$$L=\partial+\sum_{1}^{\infty} U_i\partial^{-1} \tag{1}$$

and Ψ be an eigenfunction defined to be the solution of the following simultaneous linear problems:

$$L\Psi=K\Psi \qquad \frac{\partial\Psi}{\partial y}=L_+^3\Psi \qquad \frac{\partial\Psi}{\partial t}=L_+^5\Psi \tag{2}$$

where $L_+^n=l/n$ (the differential part of L^n). Then the BKP equation is obtained as the compatibility condition of (2)

$$\frac{\partial L}{\partial y}=[L_+^3, L] \qquad \frac{\partial L}{\partial t}=[L_+^5, L]. \tag{3}$$

Exact solutions of this BKP equation were obtained in a recent communication in the framework of Krichver–Novikov theory. Here in this letter we want to discuss a dispersionless limit of the BKP equation following the methods of Kodama. For this let us set

$$X = \varepsilon x \qquad Y = \varepsilon y \qquad T = \varepsilon t \qquad V_i(X, Y, T) = U_i(x, y, z)$$

along with

$$\Psi(x, y, t) = \exp\left\{\frac{1}{\varepsilon} s(X, Y, T)\right\}.$$

It is not difficult to observe that $\partial^1\Psi \to P^1\Psi$ as $\varepsilon \to 0$ where $P = S_x$ and equations (2) become

$$K = P + \sum_{i=1}^{\infty} v_i p^{-i} \tag{4}$$

$$\frac{\partial}{\partial Y}(P\Psi) = \frac{\partial}{\partial x} Q_3\Psi = \frac{\partial}{\partial x} \lim_{\varepsilon\to 0} L_+^3\Psi = (\tfrac{1}{3} - P^3 + V_1 P)\Psi \tag{5}$$

$$\begin{aligned} &\frac{\partial}{\partial T}(P\Psi) = \frac{\partial}{\partial x}(Q_5\Psi) \\ &Q_5\Psi = \lim_{\varepsilon\to 0} L_+^5\Psi = (P^5 + V_1 P^3 + (V_3 + 2V_1^2)P + V_4). \end{aligned} \tag{6}$$

So the equations (3) become

$$\begin{aligned} &\frac{\partial P}{\partial Y} = \frac{\partial}{\partial x}[\tfrac{1}{3}P^3 + V_1 P] \\ &\frac{\partial P}{\partial T} = \frac{\partial}{\partial X}[\tfrac{1}{5} - P^5 + V_1 P^3 + (V_3 + 2V_1^2)P + V_4]. \end{aligned} \tag{7}$$

Reduced, dispersionless BKP is obtained as the compatibility of the pair (7); which leads to

$$V_{3X} = -2V_1 V_{1X} - V_{1Y} \tag{8}$$

and the required equation turns out to be

$$V_1 V_{1Y} + \partial^{-1} V_{1YY} - V_1^2 V_{1X} + V_{1X}\partial^{-1} V_{1Y} - V_{1T} = 0 \tag{9}$$

where ∂^{-1} denote $\int_{-\alpha}^{X} \mathrm{d}x$.

An exact solution to equation (9) can now be generated by use of a technique due to Kodama. Actually this method of Kodama utilizes a kind of generalized hodograph transformation. The basic theme of such a transformation is to interchange the role of dependent and independent variables. To implement this procedure we assume that the function P depends on some extra variables; $W_1 W_2, \ldots, W_m$, where $W_1 = V_1$ and $(W_1, \ldots, W_m)$ satisfy

$$\frac{\partial W_i}{\partial Y} = \sum_{j=1}^{m} A_{ij} \frac{\partial W_j}{\partial X} \qquad \frac{\partial W_i}{\partial T} = \sum_{j=1}^{m} B_{ij} \frac{\partial W_j}{\partial X}. \tag{10}$$

So, from the consistency of (10), and using equations (7) we get

$$(\nabla P)^+(A - P^2 I - V_1 I) = (P, 0, \ldots, 0) \tag{11}$$

and

$$(\nabla P)^{+}(B-P^4I-3U_1P^2I-(U_3+2U_1^2)I)$$
$$=(P^3+4U_1P_10,\dots,0)+(\nabla U_3)^t P+(\nabla U_4)^t. \tag{12}$$

Now using the fact that P and $\partial P/\partial W_1$ are independent, we deduce

$$B=A^2+U_1A+U_3I \qquad A_{11}=2U_1+\frac{\partial U_3}{\partial W_1} \qquad A_{ij}=\frac{\partial U_3}{\partial W_j} \qquad \nabla U_4=0$$

It was an important observation of Kodama that equation set (10) can be elegantly analysed if the role of dependent and independent variables are interchanged. For this it is useful to rewrite (10) in differential language from;

$$\begin{aligned} \mathrm{d}W_i \wedge \mathrm{d}T \wedge \mathrm{d}X &= \sum_{j=1}^{m} A_{ij}\,\mathrm{d}W_j \wedge \mathrm{d}Y \wedge \mathrm{d}T \\ \mathrm{d}W_i \wedge \mathrm{d}X \wedge \mathrm{d}Y &= \sum_{j=i}^{m} B_{ij}\,\mathrm{d}W_j \wedge \mathrm{d}Y \wedge \mathrm{d}T \qquad i=1,\dots,m. \end{aligned} \tag{13}$$

The writing of the differential equation in this form helps to make the interchange of dependent and independent, variables.

We now discuss different values of m separately:

$m=1$

In this case $P=P(W_1)$ where $W_1=U_1=V$

$$\frac{\partial W_1}{\partial X}=A\frac{\partial W_1}{\partial X} \qquad \text{or} \qquad \frac{\partial U}{\partial Y}=A(U)\frac{\partial U}{\partial X}. \tag{14a}$$

Similarly

$$\frac{\partial U}{\partial T}=B(U)\frac{\partial U}{\partial X} \tag{14b}$$

where $B(V)=A^2(U)+UA(U)+U_3$.

From the compatibility of (14*a*) and (14*b*) we have

$$A_T+AB_X=B_Y+BA_X. \tag{15}$$

Changing variables to W_1 we see that this equation is identically satisfied when A is a scalar function of W. We consider X to be a function of (W_1, Y, T) which are now independent variables, whence

$$\frac{\partial X}{\partial Y}=-A \qquad \frac{\partial X}{\partial T}=-B=-A^2-W_1A-U_3. \tag{16}$$

Choosing $A(U)=C$ we have $B=2U^2+U_3$ hence

$$X+CY+(C^2+CU+U_3)T=F(U).$$

Taking $F(U)=\alpha U$, we solve for V and get

$$U=\frac{1}{2T}[2CT-\alpha_{\pm}\{-2CT)^2+4(X+CY+C^2T)T\}^{1/2}]. \tag{17}$$

Furthermore in this case we can write equation (11) as

$$\frac{\mathrm{d}P}{\mathrm{d}U}(C-P^2-U)=P$$

whose solution is

$$P=\frac{-U\pm\sqrt{U^2-4(K-C)}}{2}.$$

If we set $P=\lambda$ when $U=0$, then we get

$$K=C-\lambda^2$$

whence,

$$P=-\tfrac{1}{2}U\pm\tfrac{1}{2}\sqrt{U^2+4\lambda^2}. \tag{18}$$

Now it is a very simple observation from equation (7) that the quantity P itself is a conserved quantity and hence coefficients of its expansion in any parameter (which is constant) gives us an infinite number conserved quantities. As such we have from (18)

$$P=-\frac{U}{2}+\frac{U^2}{8}-\frac{U^4}{32\lambda}+\ldots. \tag{18a}$$

So we have the following conserved quantities;

$$P_0=\frac{U^2}{8}-\frac{U}{2} \qquad P_1=-\frac{U^4}{32} \qquad \text{etc.} \tag{19}$$

$A=U$

We get $U_3=-U^2/2$, choose $F=-U^2$ so that

$$X+UY+[2U^2-U^2/2]T=-U^2$$

and we obtain

$$U=\frac{-Y\pm[Y^2-4X(2T/2+1)]^{1/2}}{(3T+2)}. \tag{20}$$

$m=2$

$$P=P(W_1,W_2) \qquad W_1=U \qquad B=A^2+UA+U_3I.$$

Let us then rewrite the compatibility condition for (10) as

$$\begin{array}{ccc} -\dfrac{\partial}{\partial W_2}(\det A-U_3) & & -\dfrac{\partial}{\partial W_2}(W_1+\operatorname{tr}A) \\ & A & \\ \dfrac{\partial}{\partial W_1}(\det A-U_3) & & \dfrac{\partial}{\partial W_1}(W_1+\operatorname{tr}A) \end{array}. \tag{21}$$

There are many structures for A which satisfy (19), we may quote two such results:

$$A=\begin{pmatrix} W_1 & 0 \\ 0 & -W_1+W_2 \end{pmatrix} \tag{22a}$$

$$A=\begin{pmatrix} -\frac{1}{2}W_1 & \frac{3}{2}W_1^2 \\ 1 & -\frac{1}{2}W_1 \end{pmatrix}. \tag{22b}$$

If we consider new dependent variables

$$X = X(W_1, W_2, T)$$
$$Y = Y(W_1, W_2, T)$$

then the hodograph equations are

$$\begin{pmatrix} -X_2 \\ X_1 \end{pmatrix} = A \begin{pmatrix} Y_2 \\ -Y_1 \end{pmatrix} \tag{23}$$

$$\begin{matrix} -\dfrac{\partial X}{\partial T}\dfrac{\partial Y}{\partial W_2} + \dfrac{\partial X}{\partial W_2}\dfrac{\partial Y}{\partial T} \\ \dfrac{\partial X}{\partial T}\dfrac{\partial Y}{\partial W_1} - \dfrac{\partial X}{\partial W_1}\dfrac{\partial Y}{\partial T} \end{matrix} = \beta \begin{pmatrix} Y_2 \\ -Y_1 \end{pmatrix}. \tag{24}$$

From the independence of Y, and ∂Y_2 we get

$$\frac{\partial Y}{\partial T} = -\mathrm{tr}\, A + U \qquad -\frac{\partial X}{\partial T} = -\det A + U_3 \tag{25}$$

so we have

$$\begin{aligned} X &= (\det A - U_3) T + F(W_1, W_2) \\ Y + (\mathrm{tr}\, A) T + W_1 T &= G(W_1, W_2) \end{aligned} \tag{26}$$

where F, G are arbitrary functions of W_1, W_2 connected by

$$\begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} = A \begin{pmatrix} G_2 \\ -G_1 \end{pmatrix}. \tag{27}$$

Now for the choice (20a) for A we get

$$F = -W_1 W_2 + W_1^2/2 + K \qquad K = \text{constant}$$

$$X = -\frac{W_1^2}{2} T + \frac{W_1^2}{2} + K + W_1 W_2 (T-1)$$

and $Y = (W_1 - W_2)T + W_2 - W_1$.

Eliminating W_2

$$\frac{W_1^2}{2}(T-1) - W_1 Y + K - X = 0 \tag{28}$$

which can be solved to yield

$$W_1 = \frac{1}{T-1}\{Y \pm [Y^2 - 2(K-X)(T-1)]^{1/2}\}. \tag{29}$$

In the present situation equation (11) can be again written as

$$P\frac{\partial P}{\partial W_1} = -1$$

$$\frac{\partial P}{\partial W_2}(-2W_1 + W_2 - P^2) = 0.$$

The second equation gives two alternatives; either

$$\frac{\partial P}{\partial W_2}=0 \qquad \text{or} \qquad P^2 = W_2 - 2W_1.$$

If $\partial P/\partial W_2 = 0$ then we easily obtain

$$P=\sqrt{K_1^2-2W_1}=K_1\left\{1-\frac{W_1}{K_1^2}-\frac{1}{2}\frac{W_1^2}{K_1^4}\dots\right\} \tag{29a}$$

which from our previous discussion again yields infinite conserved quantitites. On the other hand the second alternative $P^2 = W_2 - 2W_1$ also satisfies the first condition and due to the absence of any parameter yields only one conserved quantity.

Now we turn our attention to the second choice (20b) for A. In this case we have

$$-F_2 = A_{11}G_2 - A_{12}G_1$$

$$+F_1 = A_{21}G_2 - A_{22}G_1$$

whence $F = 3W_1/2$ and

$$X = (\det - V_3)T + F(W_1, W_2) = (-\tfrac{5}{4}W_1^2 + \tfrac{3}{2}W_1)T + \tfrac{3}{2}W_1.$$

Again we can solve for W_1 and get

$$W_1 = \tfrac{2}{5}\{\tfrac{3}{2}(T+1) \pm [\tfrac{9}{4}(T+1)^2 - 5TX)]\}. \tag{30}$$

In our above analysis we have deduced and solved the dispersion limit version of the BKP equation, the equation itself is completely integrable with an infinite number of conservation laws which are easily obtained via the expansion (equations 18a, 29a)

$$P = aK + \sum_{i=1}^{\infty} b_i K^{-i}.$$

Exact solutions are obtained via the hodographic transformation introduced by Kodama. Actually the whole procedure has more flexibility to generate many more solutions for different choices of the functions involved.

One of the authors (ND) wishes to thank CSIR (Government of India) for a fellowship.

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